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ESTIMATES IN THE KOLMOGOROV THEOREM ON CONSERVATION OF CONDITIONALLY PERIODIC MOTIONS*

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A Hamiltonian system which differs from the integrable system by a small perturbation is considered. According to the Kolmogorov theorem /1- 3/ the majority of invariant tori present in the unperturbed system do not decompose under a perturbation, and few of them become deformed. The following estimates are obtained below: for the perturbation ε under the usual conditions of nondegeneracy, the measure of the set of the decomposing tori and the deformation of the remaining tori are both estimate from above by the quantities of the order of $\sqrt{\varepsilon}$, and these estimates cannot be improved. The proof follows that /2,3/ of the Kolmogorov theorem with the intermediate estimates obtained more accurately. Similar estimates were obtained in the Moser theorem concerning the invariant curves of the mapping of a plane onto itself in /4/, and for the mapping in the multidimensional case, in /5/.

The conservation of the majority of the invariant tori was proved in /3/ also for the degenerate cases, including that of the permanent adiabatic invariance of the action variable when the Hamiltonian function varies slowly. Below it will be shown that in this case the measure of the disintegrating tori is estimate from above by a quantity of the order of $\exp(-c/\varepsilon)$ where $\varepsilon > 0$ determines the rate of change of the Hamiltonian function and $c > 0$ is a constant. The deformation of the remaining tori is estimate by a quantity of the order of ε in such a manner, that the action variable remains always in the ε-neighborhood of its initial value.

1. Formulation of the conditions and result. We shall consider a Hamiltonian system with Hamiltonian function

$$H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon) \tag{1.1}$$

where I and φ are n-dimensional vectors, ε is a small positive parameter and the function H_1 is 2π-periodic in φ. We assume that the function H, bounded region G and the positive constants $\rho, \sigma, \varepsilon_0, \vartheta, \Theta, \vartheta_1, \Theta_1, \eta, C, c, D$ satisfy the following conditions.

1°. When $\operatorname{Re} I \in G$, $|\operatorname{Im} I| < \rho$, $\operatorname{Re}\varphi \in T^n$, $|\operatorname{Im}\varphi| < \sigma$, $0 \leqslant \varepsilon < \varepsilon_0$, the function H is analytic and the inequalities $|\partial^2 H_0/\partial I^2| < \Theta$, $|H_1| < C$, $|\partial H_1/\partial I| < c$ hold. When I and φ are real, H is real. (Here T^n denotes the n-dimensional torus and $|\cdot|$ is the modulus of a complex number, norm of a vector, or a matrix).

2°. One of the following two conditions holds:

Condition of nondegeneracy. The mapping A defined by the formula $\omega = A(I) = \partial H_0/\partial I$, is a diffeomorphism of its domain of definition onto its image, and satisfies the inequalities

$$\vartheta\,|dI| \leqslant |dA| \leqslant \Theta\,|dI| \tag{1.2}$$

Condition of isoenergetic nondegeneracy. The surfaces of the level $H_0 = \text{const}$ are nonsingular: $\vartheta_1 < |\partial H_0/\partial I| < \Theta_1$. Restricting the mapping A_1 defined by the formula $A_1(I) = (\partial H_0/\partial I)/|\partial H_0/\partial I|$ to every such surface, represents a diffeomorphism of its domain of definition onto its image. The estimates (1.2) hold for dI satisfying the relation $(\partial H_0/\partial I)\,dI = 0$. When $I \in G$, then the inequality $|H_0| < \eta$ holds.

3°. The inequality $\operatorname{mes}(G \setminus G - \delta) < D\delta$, where $(G - \delta)$ is a set of points whose closed δ-neighborhoods belong to G, holds for any $\delta > 0$. In what follows, all positive quantities depending only on the constants $n, \rho, \sigma, \varepsilon_0, \vartheta, \Theta, \vartheta_1, \Theta_1, \eta, C, c, D$ introduced above, shall be called constant and denoted by C_i, c_i and a_i. The appearance of C_i in the text in some relationship is equivalent to the assertion that a constant C_i satisfying this relation exists (the same applies to the remaining constants).

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Theorem 1. For any ε and $\varkappa$ satisfying the conditions $C_1\sqrt{\varepsilon} \leqslant \varkappa < C_2^{-1}$, the set $F = G \times T^n$ can be written in the form of union of two sets, $F_\varkappa$ and $F_\varkappa'$, with the following properties.

1°. mes $F_\varkappa' < C_3\sqrt{\varepsilon}$.

2°. The set $F_\varkappa$ is a union of n-dimensional tori T_ξ of the system with the Hamiltonian (1.1). The variable ξ ordering the tori is an n-dimensional vector assuming the values from some subset of G. The torus T_ξ is defined by the following parametric equations:

$$I = \xi + f_\xi(Q), \quad \varphi = Q + g_\xi(Q), \quad Q \in T^n$$

The functions f_ξ and g_ξ are analytic on Q and satisfy the inequalities $|f_\xi| < C_4\varepsilon/\varkappa$, $|g_\xi| < C_5\varepsilon/\varkappa^2$.

3°. The motion on the torus T_ξ is conditionally periodic and defined by the formula $Q^\cdot = \omega_\xi$, and the frequency vector ω_ξ satisfies the inequalities $|(\omega_\xi, k)| \geqslant \varkappa |k|^{-n}$ for all enumerable vectors $k \neq 0$. The theorem is proved in Sect.2.

Setting in the theorem 1 $\varkappa = C_1\sqrt{\varepsilon}$, we obtain the following result.

Corollary 1. When $0 < \varepsilon < C_6^{-1}$, we can write the set F in the form $F = U \cup U'$ where mes $U' < C_7\sqrt{\varepsilon}$ and the set U is the union of the n-dimensional invariant tori T_ξ along every one of which $|I - \xi| < C_8\sqrt{\varepsilon}$ for some $\xi \in R^n$.

Thus the measure of the set of tori decomposing under a perturbation is of the order $O(\sqrt{\varepsilon})$. Every invariant torus belonging to the set U differs from a certain invariant torus of the unperturbed problem $I = \xi = \text{const}$ by a deformation of the order $O(\sqrt{\varepsilon})$. The tori belonging to U deform differently. According to Theorem 1 the measure of the set consisting of the invariant tori deformed by more than $\varepsilon/\varkappa$ is $O(\varkappa)$. Examples of a pendulum in a weak gravity field ($H = {}^1/_2 I^2 - \varepsilon\cos\varphi$) show that the above estimates cannot be improved.

Note. Analogous estimates were obtained in /5/ for the symplectic, sufficiently smooth mapping close to integrable. Using the results of /5/, we can reduce the Hamiltonian system to a symplectic mapping and thus obtain the estimates of the corollary 1 for the case of isoenergetic nondegeneracy (for $n = 2$ the estimates can be obtained using the results of /4/). Moreover, the results imply that the invariant tori of the perturbed system can be included in the smooth family of tori.

Let us now consider the case of two degrees of freedom, assuming that the condition of isoenergetic nondegeneracy holds. In this case the two-dimensional invariant tori divide the three-dimensional energy level $H = \text{const}$, and the conservation of the majority of tori implies that the values of the variables I along the motion will always remain close to their initial values /2/, whatever they are. Theorem 1 implies that in this case the sizes of the gaps between the tori and the deformation of each torus are of the order $O(\sqrt{\varepsilon})$, and this leads us to the following assertion.

Corollary 2. Let a system with two degrees of freedom be isoenergetically nondegenerated and let the conditions of Theorem 1 hold. Then the inequality

$$|I(t) - I_0| < C_{11}\sqrt{\varepsilon}, \quad -\infty < t < \infty$$

holds along the motion for $0 < \varepsilon < C_9^{-1}$ and all initial data $(I_0, \varphi_0) \in (G - C_{10}\sqrt{\varepsilon}) \times T^2$

Example 1. /3/. We consider a plane, circular, bounded three-body problem. Let the sun be of mass 1, and Jupiter of mass ε. In accordance with Corollary 2 the oscillations of the semiaxis and the excentricity of the asteroid are of the order $O(\sqrt{\varepsilon})$ (provided that the unperturbed orbit of the asteroid does not intersect the orbit of Jupiter).

Example 2. /2/. We consider the rotation of a heavy rigid body about a fixed point. Let ε be the ratio of the difference between the largest and the smallest value of the potential energy of the body to its kinetic energy at the initial instant.* Then, during the whole motion, the relative oscillations in the value of the kinetic moment vector modulus and the oscillations in the value of the angle between this vector and the vertical will be of the order $O(\sqrt{\varepsilon})$ (with the initial data at a distance from the separatrices of the Euler–Poinsot problem).

2. Proof of Theorem 1.

The proof is carried out, for definiteness, under the assumption that the condition of nondegeneracy given in Sect.1, holds. The construction follows, on the whole, that given in /2,3/, with the intermediate estimates changed. It is assumed that the norm $|\cdot|$ is given by the formula $|X| = \max_{i,j}|x_{i,j}|$ with $X = (x_{i,j})$.

2.1. Auxilliary assertions. Lemma 1. The following sequences appear in the conditions of Theorem 1 when $0 < \varepsilon < \varepsilon_0$, $\varkappa \geqslant c_1\sqrt{\varepsilon}$: the sequence of positive numbers $\beta_s, \gamma_s, M_s, N_s$, the sequence of imbedded regions $V^{(s)}$ in the space I, and $W^{(s)}$ in the space (I, φ), the sequence of canonical diffeomorphisms $B^{(s)}$ mapping $W^{(s)}$ onto $W^{(s-1)}$, the sequence of determined and analitical functions $H^{(s)} = H_0^{(s)}(I) + H_1^{(s)}(I, \varphi)$ when $(I, \varphi) \in W^{(s)}$ and the sequence of diffeomorphism $A^{(s)}$ mapping $V^{(s)}$ onto the space of frequencies ω. These sequences have the following properties: 1°. The numerical sequences are constructed as follows:

$$\gamma_s = 2^{-(s+1)}\sigma, \quad M_0 = C\varepsilon, \quad M_s = \frac{c_3 2^s M_{s-1}^2}{\varkappa^2\gamma_s^{2(2n+1)}}\left|\ln\frac{c_2M_{s-1}}{\varkappa^2}\right|^n, \quad s \geqslant 1, \quad N_s = \frac{4}{\gamma_s}\left|\ln\frac{c_2M_{s-1}}{\varkappa^2}\right|, \quad \beta_s = 2^{-s}\vartheta^{-1}\varkappa N_s^{-n}$$

2°. The regions $V^{(s)}$ and $W^{(s)}$ are constructed as follows:

$$V^{(0)} = \{I : \operatorname{Re} I \in G, \ |\operatorname{Im} I| < \rho\}, \ W^{(0)} = \{I, \varphi : I \in V^{(0)}, \ |\operatorname{Im}\varphi| < \sigma\}$$

$$\bar V^{(s)} = \left\{I : I \in V^{(s-1)}, \ \left|\left(k, \frac{\partial H_0^{(s-1)}}{\partial I}\right)\right| > \varkappa|k|^{-n}, \ k \in Z^n, \ 1 \leqslant |k| \leqslant N_s\right\}, \ V^{(s)} = \bar V^{(s)} - \beta_s$$

$$W^{(s)} = \{I, \varphi : I \in V^{(s)}, \ |\operatorname{Im}\varphi| < \sigma - \sum_{i=1}^{s}\gamma_i\}, \quad s \geqslant 1$$

3°. The canonical analytic diffeomorphism $B^{(s)}: (I^{(s)}, \varphi^{(s)}) \to (I^{(s-1)}, \varphi^{(s-1)})$ maps $W^{(s)}$ onto $W^{(s-1)}$ so that the following inequalities hold:

$$|I^{(s)} - I^{(s-1)}| < \frac{c_4M_{s-1}}{\varkappa\gamma_s^{2n+1}} < \frac{\beta_s}{2}$$

$$|\varphi^{(s)} - \varphi^{(s-1)}| < \frac{c_4M_{s-1}}{\varkappa\beta_s\gamma_s^{2n}} < \frac{\gamma_s}{2}$$

$$|dI^{(s-1)}| \leqslant \left(1 + \frac{c_5M_{s-1}}{\varkappa\beta_s\gamma_s^{2n+1}}\right)|dI^{(s)}| + \frac{c_5M_{s-1}}{\varkappa\gamma_s^{2n+2}}|d\varphi^{(s)}|$$

$$|d\varphi^{(s-1)}| \leqslant \frac{c_5M_{s-1}}{\varkappa\beta_s^2\gamma_s^{2n}}|dI^{(s)}| + \left(1 + \frac{c_5M_{s-1}}{\varkappa\beta_s\gamma_s^{2n+1}}\right)|d\varphi^{(s)}|$$

For the case $s = 1$ the second inequality can be sharpened

$$|\varphi^{(1)} - \varphi^{(0)}| < \frac{c_6M_0}{\varkappa^2\gamma_1^{3(n+1)}}$$

The diffeomorphism $B^{(s)}$ transforms real points into real points.

4°. The Hamiltonian $H^{(s)}$ is given by the formulas

$$H^{(0)}(I, \varphi) = H(I, \varphi), \ H^{(s)}(I, \varphi) = H(B^{(1)} \circ B^{(2)} \circ \ldots \circ B^{(s)}(I, \varphi)), \quad s \geqslant 1$$

and satisfies the inequality $|H_1^{(s)}| < M_s$.

5°. The mapping $A^{(s)}$ given by the formula

$$A^{(s)}(I) = \partial H_0^{(s)}(I)/\partial I$$

is a diffeomorphism of $V^{(s)}$ onto $A^{(s)}(V^{(s)})$ and satisfies the inequalities

$$1/2\,\vartheta\,|dI| \leqslant |dA^{(s)}| \leqslant 2\Theta\,|dI|$$

The sequences of direct $A^{(s)}$ and inverse $A^{(s)^{-1}}$ mapping satisfy the inequalities

$$A^{(s)}(V^{(s)}) \subset A^{(s-1)}(\bar V^{(s)}) - {}^1/_4\vartheta\beta_s, \quad |A^{(s)} - A^{(s-1)}| < \frac{M_{s-1}}{\beta_s}, \quad |A^{(s)^{-1}} - A^{(s-1)^{-1}}| < \frac{4M_{s-1}}{\vartheta\beta_s}$$

The proof of Lemma 1 follows directly from Lemmas 2 and 3 given below.

Lemma 2. Let the region $V \subset C^n$, the function $\Phi(I, \varphi) = \Phi_0(I) + \Phi_1(I, \varphi)$ and the number M have the following properties.

1°. Function Φ is analytic in the region

$$W = \{I, \varphi : I \in V, \ |\operatorname{Im}\varphi| < \sigma_1\}, \quad 1/2\sigma < \sigma_1 \leqslant \sigma$$

and satisfies the estimate $|\Phi_1| < M$. The function Φ_0 and Φ_1 are real in $\operatorname{Re} W$.

2°. The mapping A given by the formula

$$A(I) = \partial\Phi_0(I)/\partial I$$

is a diffeomorphism of V onto $A(V)$ and satisfies the inequalities

$$\tfrac{1}{2}\vartheta\,|dI| \leqslant |dA| \leqslant 2\Theta\,|dI|$$

Then for any positive numbers $\varkappa, s, \beta, \gamma, N$ connected by the conditions

$$\frac{M}{\varkappa\beta\gamma^{2n+1}} < c_7^{-1}, \quad \frac{M}{\beta^2} < c_8^{-1}, \quad \beta = 2^{-s}\vartheta^{-1}\varkappa N^{-n}, \quad N = \frac{4}{\gamma}\left|\ln\frac{c_2 M}{\varkappa^2}\right| \tag{2.1}$$

the following relations hold.

1) We define the regions $\bar V$, V' and W' by the relations

$$\bar V = \{I: I \in V, \ |(k, \partial\Phi_0/\partial I)| > \varkappa|k|^{-n}, \ k \in Z^n, \ 1 \leqslant |k| \leqslant N\}$$
$$V' = \bar V - \beta, \quad W' = \{I, \varphi: I \in V', \ |\operatorname{Im}\varphi| < \sigma_1 - \gamma\}$$

A canonical analytic diffeomorphism $B: J, \psi \to I, \varphi$ exists, mapping W' onto W and satisfying the inequalities

$$|J - I| < \frac{c_4 M}{\varkappa\gamma^{2n+1}} < \frac{\beta}{2}, \quad |\psi - \varphi| < \frac{c_4 M}{\varkappa\beta\gamma^{2n}} < \frac{\gamma}{2} \tag{2.2}$$
$$|dI| \leqslant \left(1 + \frac{c_5 M}{\varkappa\beta\gamma^{2n+1}}\right)|dJ| + \frac{c_5 M}{\varkappa\gamma^{2n+2}}|d\psi|$$
$$|d\varphi| \leqslant \frac{c_5 M}{\varkappa\beta^2\gamma^{2n}}|dJ| + \left(1 + \frac{c_5 M}{\varkappa\beta\gamma^{2n+1}}\right)|d\psi|$$

If we also put $|\partial\Phi/\partial I| < M$, then

$$|\psi - \varphi| < \frac{c_6 M}{\varkappa^2\gamma^{3(n+1)}}$$

The diffeomorphism B transforms real points into real points.

2) We write the function $\Phi'(I, \varphi) = \Phi(B(I, \varphi))$ in the form

$$\Phi'(I, \varphi) = \Phi_0'(I) + \Phi_1'(I, \varphi)$$
$$\Phi_0'(I) = \Phi_0(I) + (2\pi)^{-n}\int_0^{2\pi}\cdots\int_0^{2\pi}\Phi_1(I, \varphi)\,d\varphi, \quad \Phi_1' = \Phi' - \Phi_0'$$

The following inequality holds:

$$|\Phi_1'| < \frac{c_3 2^s M^2}{\varkappa^2\gamma^{2(2n+1)}}\left|\ln\frac{c_2 M}{\varkappa^2}\right|^n \tag{2.3}$$

3) The mapping A' defined by the formula

$$A'(I) = \partial\Phi'_0(I)/\partial I$$

is a diffeomorphism of V' onto $A'(V')$ and satisfies the inequalities

$$A'(V') \subset A(\bar V) - \tfrac{1}{4}\vartheta\beta, \quad |A' - A| < \frac{M}{\beta}, \quad |A'^{-1} - A^{-1}| < \frac{4M}{\vartheta\beta}, \quad |dA' - dA| \leqslant \frac{2nM}{\beta^2}|dI|$$

Proof. Consider the Fourier series for the function Φ_1

$$\Phi_1 = \bar\Phi_1(I) + \Phi_{1N}(I, \varphi) + R_N(I, \varphi), \quad \Phi_{1N} = \sum_{1\leqslant|k|\leqslant N} f_k(I)\exp(i(k, \varphi)), \quad R_N = \sum_{|k|>N} f_k(I)\exp(i(k, \varphi))$$

and the canonical change of variables $I, \varphi \to J, \psi$ defines by the formulas

$$I = J + \frac{\partial G(J, \varphi)}{\partial\varphi}, \quad \psi = \varphi + \frac{\partial G(J, \varphi)}{\partial J}, \quad G(I, \varphi) = \sum_{1\leqslant|k|\leqslant N}\frac{if_k(I)\exp(i(k, \varphi))}{(k, \partial\Phi_0(I)/\partial I)} \tag{2.4}$$

where the number N remains, for the time being, arbitrary. Following /3/, we prove the following assertion. Let the positive numbers $\varkappa, \beta, \gamma$ satisfy the first inequality of (2.1). Then the formulas (2.4) determine for $(J, \psi) \in W'$ the canonical analytic diffeomorphism $B: W' \to W$ satisfying the inequalities (2.2).

Let us perform a change of variables in the Hamiltonian system in question according to the formulas (2.4). The change in the new variables will be described by the Hamiltonian

$$\Phi'(J, \psi) = \Phi_0(J) + \left[\frac{\partial\Phi_0}{\partial J}\frac{\partial G}{\partial\varphi} + \Phi_1(J, \varphi)\right] + \left[\Phi_0\left(J + \frac{\partial G}{\partial\varphi}\right) - \Phi_0(J) - \frac{\partial\Phi_0}{\partial J}\frac{\partial G}{\partial\varphi}\right] + \left[\Phi_1\left(J + \frac{\partial G}{\partial\varphi}, \varphi\right) - \Phi_1(J, \varphi)\right] \tag{2.5}$$

(in the right-hand part of (2.5) φ must be expressed in terms of ψ according to (2.4)). By virtue of the definition of G (2.4) the term in the first square brackets in equal to $\overline{\Phi}_1(J)+R_N(J,\varphi)$. Let us write

$$\Phi_0'(J)=\Phi_0(J)+\overline{\Phi_1}(J),\quad \Phi_1'(J,\psi)=\Phi'(J,\psi)-\Phi_0'(J)$$

Estimating the right-hand side of (2.5) we obtain, as in /3/,

$$|\Phi_1'|<\frac{\theta c_4{}^2n^2M^2}{\varkappa^2\gamma^{2(2n+1)}}+\frac{2c_4nM^2}{\varkappa\beta\gamma^{2n+1}}+\frac{a_1M}{\gamma^n}e^{-{}^1/_4\gamma N}$$

(here we utilize the Cauchy estimate for $\partial\Phi_1/\partial J$ and the estimate for the last term of the Fourier series given in /3/).

Choosing $c_2=\theta c_4{}^2, N, \beta$ according to (2.1), we obtain the inequality (2.3). The definition of A' and the Cauchy inequality yield, for $I\in V'$,

$$|A(I)-A'(I)|=|\partial\overline{\Phi}_1/\partial I|<M\beta^{-1}$$
$$|dA-dA'|=|(\partial^2\overline{\Phi_1}/\partial I^2)\,dI|\leqslant 2nM\beta^{-2}|dI|$$

The remaining inequalities of Lemma 2 follow from the "Lemma on the frequency variation" /3/.

Lemma 3. Let us consider the sequence M_s of Lemma 1. When $\varkappa\geqslant 2c_9\sqrt{\varepsilon}$, the following inequality holds:

$$M_s<c_{10}\varepsilon\left(\frac{c_9{}^2\varepsilon}{\varkappa^2}\right)^{r-1},\quad r=\left(\frac{3}{2}\right)^s$$

Proof. We define the number $a_1>0$ by the condition that the inequality $|\ln z|^n<z^{-1/2}$ holds for $0<z<a_1{}^{-1}$. We shall show that $a_2>0$ can be chosen such, that when $\varkappa>a_2\sqrt{\varepsilon}$, the inequality

$$c_2M_i\varkappa^{-2}<a_1{}^{-1}\tag{2.6}$$

holds. When $\varkappa>(a_1c_2C\varepsilon)^{1/2}$, the inequality (2.6) holds for $i=0$. Let us assume that it also holds for $0\leqslant i\leqslant s-1$. Using the definition of M_i and γ_i we obtain, for $1\leqslant i\leqslant s$,

$$M_i<\frac{c_3 2^i M_{i-1}^2}{\varkappa^2\gamma_i^{2(2n+1)}}\left(\frac{\varkappa^2}{c_2M_{i-1}}\right)^{1/2}=\frac{a_3a_4{}^iM_{i-1}^{3/2}}{\varkappa}$$

and this yields.

$$M_s<\frac{a_3a_4{}^s}{\varkappa}\cdot\left(\frac{a_3a_4^{s-1}M_{s-2}^{3/2}}{\varkappa}\right)^{3/2}=\frac{a_3^{1+3/2}a_4^{s+3/2(s-1)}M_{s-2}^{(3/2)^2}}{\varkappa^{1+3/2}}<\frac{a_3^{2(r-1)}a_4^{6r-2s-6}}{\varkappa^{2(r-1)}}M_0{}^r<a_5M_0\left(\frac{a_6M_0}{\varkappa^2}\right)^{r-1},\quad r=\left(\frac{3}{2}\right)^s$$

Since $M_0=C\varepsilon$, we have

$$M_s<c_{10}\varepsilon\left(\frac{a_7\varepsilon}{\varkappa^2}\right)^{r-1},\quad \frac{c_4M_s}{\varkappa^2}<a_8\left(\frac{a_7\varepsilon}{\varkappa^2}\right)^r\tag{2.7}$$

It is now clear that for a sufficiently large a_2 and $\varkappa>a_2\sqrt{\varepsilon}$ the inequality (2.6) holds also for $i=s$. By induction we now find that for $\varkappa>a_2\sqrt{\varepsilon}$ the inequalities (2.6) and (2.7) hold for all i,s and the inequality of Lemma 3 follows from (2.7).

2.2. Derivation of Theorem 1 from Lemmas 1 and 3. Let

$$C_1\sqrt{\varepsilon}\leqslant\varkappa<C_2^{-1},\quad C_1=\max(c_1,2c_9),\ C_2=\max(2\theta^{-1}\rho^{-1},\quad C_1^{-1}\varepsilon_0^{-1/2})$$

We consider the objects $\beta_s, \gamma_s, M_s, N_s, V^{(s)}, W^{(s)}, B^{(s)}, H^{(s)}, A^{(s)}$ defined in Lemma 1, and put

$$V^{(\infty)}=\bigcap_{i=0}^{\infty}V^{(i)},\quad W^{(\infty)}=\bigcap_{i=0}^{\infty}W^{(i)}$$

Following /3/ we show that the sequence of canonical diffeomorphisms $S^{(i)}=B^{(1)}\circ B^{(2)}\circ\ldots\circ B^{(i)}$ converges uniformly on the set $W^{(\infty)}$ to some continuous 1:1 mapping $S^{(\infty)}$, the sequence of the Hamiltonians $H^{(i)}(I,\varphi)$ converges to the Hamiltonian $H^{(\infty)}(I)$ independent of the phase φ, and the sequence of diffeomorphisms $A^{(i)}$ converges on $V^{(\infty)}$ to a continuous 1:1 mapping $A^{(\infty)}$. We can show, as in /3/, that $\operatorname{mes}(F\setminus W^{(\infty)})<C_3\varkappa$. Let us write $F_\varkappa=\operatorname{Re}S^{(\infty)}(W^{(\infty)})$, $F_\varkappa'=F\setminus F_\varkappa$. Since all $S^{(i)}$ preserve the measure, we have

$$\operatorname{mes}F_\varkappa=\operatorname{mes}S^{(\infty)}(W^{(\infty)})=\operatorname{mes}\bigcap_i S^{(i)}(W^{(i)})=\operatorname{mes}\bigcap_i W^{(i)}=\operatorname{mes}W^{(\infty)}$$

$$\operatorname{mes}F_\varkappa'=\operatorname{mes}F\setminus F_\varkappa=\operatorname{mes}F\setminus W^{(\infty)}<C_3\varkappa$$

(we take the measure of the real component of a set as the measure of this set).

Further, following /3/ we show that the initial Hamiltonian H determines, in the variables ξ, Q defined by the change of variables $(I, \varphi) = S^{(\infty)}(\xi, Q)$ the motion $\xi = \text{const}$, $Q^{\cdot} = A^{(\infty)}(\xi)$ on $F_\varkappa$. This means that the set $F_\varkappa$ consists of the n-dimensional tori of the system with Hamiltonian (1.1). Every such torus can be specified either by the value of $\xi \in \operatorname{Re} V^{(\infty)}$, or of $\omega = A^{(\infty)}(\xi) \in A^{(\infty)}(V^{(\infty)})$, $|(k, \omega)| \geqslant \varkappa |k|^{-n}$, $|k| \neq 0$, $k \in Z^n$. Finally, from Lemmas 1 and 3 it follows that the change $(I, \varphi) = S^{(\infty)}(\xi, Q)$, is analytic in Q and satisfies the inequalities

$$|I - \xi| < C_4 \frac{\varepsilon}{\varkappa}, \quad |\varphi - Q| < C_5 \frac{\varepsilon}{\varkappa^2}$$

Q.E.D.

3. System with two degrees of freedom in the case of self-degeneracy.

Consider a Hamiltonian system with the following Hamiltonian function:

$$H(I, \varphi, \varepsilon) = H_0(I_0) + \varepsilon H_{10}(I) + \varepsilon H_{11}(I, \varphi) + \varepsilon^2 H_2(I, \varphi, \varepsilon) \tag{3.1}$$

$$I = (I_0, I_1), \quad \varphi = (\varphi_0, \varphi_1), \quad \int_0^{2\pi} H_{11}\, d\varphi_0 = 0$$

where ε is a small positive parameter and the function H is (2π)-periodic in φ_0 and φ_1. As we know /3/, in the general case of a system with two degrees of freedom degenerated once, the Hamiltonian is reduced to the form (3.1). We assume that the function H, the positive constants $\rho, \sigma, \varepsilon_0, \vartheta, \Theta, \vartheta_1, C, c, a_0, b_0, a_1, b_1$ and the rectangular region $G = (a_0, b_0) \times (a_1, b_1)$ are such, that when $\operatorname{Re} I \in G$, $|\operatorname{Im} I| < \rho$, $\operatorname{Re} \varphi \in T^2$, $|\operatorname{Im} \varphi| < \sigma$, $0 \leqslant \varepsilon < \varepsilon_0$, the function H is analytic and the following inequalities hold:

$$|H_{11}| + |H_2| < C, \ |\partial H_{11} / \partial I| < c$$

$$\left|\frac{\partial^2 H_0}{\partial I_0^2}\right| + \left|\frac{\partial^2 H_{10}}{\partial I^2}\right| < \Theta, \quad \left|\frac{\partial H_0}{\partial I_0}\right| > \vartheta, \quad \left|\frac{\partial H_{10}}{\partial I_1}\right| > \vartheta, \quad \left|\frac{\partial^2 H_{10}}{\partial I_1^2}\right| > \vartheta_1$$

When I and φ are real, H is real.

Theorem 2. Positive constants $C_1, C_2, \ldots, C_7$ exists such that when $0 < \varepsilon < C_1^{-1}$, the set $F = (G - C_2\varepsilon) \times T^2$ can be represented as a union of two sets (F_ε and F_ε') with the following properties:

1°. $\operatorname{mes} F_\varepsilon' < C_3 \exp(-C_4^{-1}/\varepsilon)$.

2°. The set F_ε is a union of two-dimensional invariant tori T_ξ of the system with the Hamiltonian (3.1). The variable ξ enumerating the tori is a two-dimensional vector assuming the values from some subset of G. The torus T_ξ is given in the parametric form by the equation

$$I = \xi + f_\xi(Q), \ \varphi = Q + g_\xi(Q), \ Q \in T^2$$

The functions f_ξ and g_ξ are analytic in Q and satisfy the inequalities

$$|f_\xi| < C_5\varepsilon, \ |g_\xi| < C_6\varepsilon$$

3°. The formula $Q^{\cdot} = \omega_\xi$ gives the motion on the torus T_ξ, and the frequency vector $\omega_\xi = (\omega_0, \varepsilon\omega_1)$ satisfies the inequalities $|(\omega_\xi, k)| > (\exp(-C_4^{-1}/\varepsilon))\, |k|^{-2}$ for all enumerable vectors $k \neq 0$.

4°. The following inequality holds along the motion for any initial values $(I(0), \varphi(0)) \in F$:

$$|I(t) - I(0)| < C_7\varepsilon, \ -\infty < t < \infty$$

The equation of motion which can reduced to a system with the Hamiltonian (3.1), appear in the problems of permanent adiabatic invariance of the action variables, in the vibrational system with one degree of freedom where the Hamiltonian undergoes a slow periodic variation, and in the vibrational system with two degrees of freedom when the Hamiltonian function depends smoothly on one of the coordinates /3/. Theorem 2 implies that in the above problems, under the normal assumption concerning the nondegeneracy /3/, only a part of the phase space of the order $O(\exp(-c/\varepsilon))$, $c = \text{const}$ may remain unfilled by the invariant tori. The variation of the action variables is bounded by a quantity of the order of ε, where ε characterizes the smoothness with which the Hamiltonian depends on time or on a specified coordinate.

Example 3 /3/. We consider the motion of a charged particle in an axially symmetric magnetic trap. Let the ratio of the initial radius of the Larmor spiral of the particle to the characteristic dimension of the trap be equal to ε, and the ratio of this radius to the initial pitch of the spiral be of the order of unity. Then, during the whole motion, the relative oscillations of the magnetic moment of the particle are bounded by a quantity of the order ε. The position of the magnetic mirror may differ from that computed according to the adiabatic theory only by a quantity of the order of the Larmor radius.

We shall now describe a scheme for proving Theorem 2. The system (3.1) contains one rapid variable, namely the angle φ_0. The classical perturbation theory allows us to use an almost identical canonical change of variables in order to eliminate the dependence of the Hamiltonian on φ_0, in any finite order of ε. More accurate estimates show /6/ that the dependence ε can be transferred to the exponentially small terms. An analytic canonical variable change $(I, \varphi) \to (J, \psi)$ reduces the Hamiltonian to the form

$$H = H_0(J_0) + \varepsilon H_{10}(J) + \varepsilon^2\Phi_2(J, \psi_1, \varepsilon) + \Phi_3(J, \psi, \varepsilon) \tag{3.2}$$
$$|I - J| + |\varphi - \psi| = O(\varepsilon),\ J = (J_0, J_1),\ \psi = (\psi_0, \psi_1)$$
$$\Phi_3 = O(\exp(-c_1^{-1}/\varepsilon)),\ c_1 = \text{const},\ c_1 > 0$$

The system with the Hamiltonian $H_0 + \varepsilon H_{10} + \varepsilon^2\Phi_2$ can be integrated with help of the canonical variable change $(J, \psi) \to (J', \psi')$, and

$$|J - J'| + |\psi - \psi'| = O(\varepsilon),\quad J' = (J_0', J_1',), \psi' = (\psi_0', \psi_1'),\ J_0' = J_0 \tag{3.3}$$

Carrying out this change in (3.2), we reduce the Hamiltonian to the form

$$H = H_0(J_0') + \varepsilon\Psi_0(J', \varepsilon) + \Psi_1(J', \psi', \varepsilon) \tag{3.4}$$
$$\Psi_1 = O(\exp(-c_1^{-1}/\varepsilon))$$

Let us now consider the Hamiltonian (3.4) as a perturbation of the Hamiltonian $H_0(J'_0) + \varepsilon\Psi_0(J', \varepsilon)$. Formally, the condition of Theorem 1 are not fulfilled in this case since the principal part of the Hamiltonian depends on ε, but the statement of the theorem itself remains valid and can be proved as in Sect.2. It follows therefore that the phase space of the system (3.4) with the exception of an exponentially small measure, is filled with the invariant tori differing exponentially little from the tori $J' = \text{const}$. Combining this argument with the inequalities (3.2) and (3.3), we arrive at the proof of Theorem 2.

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REFERENCES

1. KOLMOGOROV A.N., On the conservation of conditionally periodic motions under a small variation of the Hamilton function. Dokl. Akad. Nauk SSSR, Vol.98, No.4, 1954.
2. ARNOL'D V.I., Proof of the Kolmogorov theorem on the conservation of conditionally periodic motions under a small variation of the Hamilton function. Uspekhi matem. nauk, Vol.18, No.5, 1963.
3. ARNOL'D V.I., Small denominators and the problems of stability in the classical and celestial mechanics. Uspekhi matem. nauk, Vol.18, No.6, 1963.
4. LAZUTKIN V.F., On the Moser theorem on invariant curves. In book: Problems of Dynamic Theory of Propagation of Seismic Waves. Ed.14, Leningrad, Izd. LGU, 1974.
5. SVANIDZE N.V., Small perturbations in an integrable dynamic system with an integral invariant. In book: Boundary Value Problems of Mathematical Physics. 10. Tr. MIAN, SSSR, Vol. 147, 1980.
6. NEISHTADT A.I., On the accuracy of conservation of the adiabatic invariant. PMM, Vol.45, No.1, 1981.

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